

Quantum spacetime + gauge symmetry = metric signature

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References: [1] - arXiv:2503.07176 , [2] - arXiv:0901.2710

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Introduction

Noncommutative geometry in physics is used to model "manifolds" with intrinsic minimal length scales. Often it is incorporated by deforming the algebra of smooth spacetime functions $C^\infty(M)$ by systematically promoting the pointwise function product $\cdot$ into a noncommutative product $\star$. This $\star$ product imposes noncommutativity of coordinate functions $[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu \neq 0$ and propagates to an uncertainty relation of position measurement. As a mathematical field, noncommutative geometry studies geometric constructions on noncommutative algebraic structures (noncommutative algebras and modules over them).

In this presentation we show how deformations which impose (almost Abelian) Lie algebraic commutation relations on coordinates satisfy a unique property that $U(N)$ gauge invariance (of a natural choice of an action) is possible only on specific metric signatures for a given Lie algebraic deformation.

1 - Deformed product

Consider an almost abelian Lie group

$$\mathcal{G} = \mathbb{R} \ltimes_a \mathbb{R}^n \quad a \in \text{Mat}_{n \times n}$$

whose group product is given as [1]

$$(p^M, \vec{p}) (q^M, \vec{q}) = (p^M + q^M, \vec{p} + a(p^M) \vec{q})$$

for $p^M, q^M \in \mathbb{R}$ and $\vec{p}, \vec{q} \in \mathbb{R}^n$. Then, for $A(p^M) = a(p^M) \oplus 1_{1 \times 1} \in \text{Mat}_{n+1 \times n+1}$ one can consider a star product on the $n+1$ dimensional spacetime as

$$(f \star g)(x) = \frac{1}{(2\pi)} \int dp^M dy^M e^{-ip^M y^M} f(x + y^M) g(A(p^M)x)$$

This deformed product imposes the commutators on coordinates as

$$[x^M, x^\mu]_\star = -i [\partial_{p^M} A(p^M)|_{p^M=0}]^\mu \sigma x^\sigma.$$

2 - Multi-derivation structure

It turns out that partial derivatives form twisted multi-derivations [2] of the deformed algebra $C_\star^\infty(M)$ in the sense

$$\partial_\mu(f \star g) = (\partial_\mu f) \star g + A(P_M)^\rho{}_\mu(f) \star \partial_\rho g$$

where $P_M = -i \frac{\partial}{\partial x^M}$

3 - Gauge structure

In noncommutative geometry one can define a connection over a noncommutative algebra's right module M . M is a right module over an algebra A is there exists a map $\triangleleft: A \rightarrow \text{Aut}(M)$ which obeys

$$(m \triangleleft a) \triangleleft b = (m) \triangleleft (ab), \forall a, b \in A, m \in M$$

The module M models sections of vector bundles, as commutatively they form a right module over spacetime functions $C^\infty(M)$ ($(\Psi \triangleleft f)(x) = \Psi(x) \cdot f(x)$). On M one can define a covariant derivative in the direction of a derivation X (**let us for now forget that partial derivatives are not derivations of the algebra in our almost abelian case**) of the algebra A as any map $\nabla_X: M \rightarrow M$ satisfying the Leibniz rule

$$\nabla_X(m \triangleleft a) = m \triangleleft X(a) + \nabla_X(m) \triangleleft a \quad (1)$$

where derivations X of the algebra A are linear maps $X: A \rightarrow A$ satisfying

$$X(ab) = X(a)b + aX(b)$$

Gauge transformations are defined as automorphisms of left module M , i.e., invertible linear maps ϕ which are also right A -linear

$$\phi(m \triangleleft a) = \phi(m) \triangleleft a$$

This exactly reproduces the $C^\infty(M)$ linearity of gauge transformations in the commutative case.

With that said, $\nabla^\phi := \phi \circ \nabla_X \circ \phi^{-1}$ satisfies the same Leibniz rule as ∇_X . Additionally, it satisfies $\phi \nabla_X(m) = \nabla^\phi(\phi m)$. One can define curvature of the connection as the right A -linear map for any two derivations X, Y as

$$F(X, Y)(m) = [\nabla_X, \nabla_Y](m) - \nabla_{[X, Y]}(m).$$

It can be shown that $F^\phi(X, Y) = \phi \circ F(X, Y) \circ \phi^{-1}$ is the curvature of ∇_X^ϕ .

To model electromagnetism's $U(1)$ connection, we actually choose the module to equal the algebra itself and we restrict to gauge transformations given as multiplication by unitary elements g . One can show, via the Leibniz rule, that any connection is completely determined by its action on the unit element of the algebra

$$\begin{aligned} \nabla_X(a) &= \nabla_X(1 \star a) = 1 \star X(a) + \nabla_X(1) \star a \\ &= X(a) + i A_X \star a \end{aligned}$$

where A_X can be thought of the contraction of the vector field X and gauge potential A . Similarly, curvature's components come from acting on the identity $F_{\mu\nu} \equiv F(\partial_\mu, \partial_\nu)(1)$ and we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]_\star$$

Using this curvature, we define an action principle

$$S = \int d^4x F_{\mu\nu} \star F_{\rho\sigma}^\dagger \star g^{\mu\rho} \star g^{\nu\sigma}$$

Results

A complication that we encountered is that the Leibniz rule (1) is well defined only for derivations X , but, as we have announced in Section 2, partial derivatives are actually twisted multi-derivations, not derivations. Thus, the concept of the connection needs to be adapted. (1) is impossible to consistently satisfy, but, the following Leibniz rule

$$\nabla_\mu(a \star f) = A(\lambda P_M)^\rho{}_\mu(a) \star \partial_\rho(f) + \nabla_\mu(a) \star f \quad (2)$$

is satisfied by connections parametrized by $A_\mu = \nabla_\mu(1)$ in the following usual way

$$\nabla_\mu(a) = \partial_\mu(a) + i A_\mu \star a. \quad (3)$$

Additionally, for a unitary automorphism $\phi_g(a) = g \star a$ the concept of gauge transformations needs to be deformed as well. The correct choice amounts to

$$\nabla_\mu^g(a) = A(\lambda P_M)^\rho{}_\mu(g) \star \nabla_\rho(g^\dagger \star a) \quad (4)$$

which affects the transformation rules of the potential

$$A_\mu^g = \nabla_\mu^g(1) = A(P_M)^\rho{}_\mu(g) \star (\partial_\rho g^\dagger + A_\rho \star g^\dagger)$$

and of the curvature

$$F_{\mu\nu}^g = g \star F_{\alpha\beta} \star A^{-1}(\lambda P_M)^\alpha{}_\mu A^{-1}(\lambda P_M)^\beta{}_\nu (g^\dagger)$$

The condition for gauge invariance $S^g = S$ gives finally a condition on $A(\lambda P_M)$ and $g^{\mu\nu}$ as a relation which has hold for all unitary elements $h \in C_\star^\infty(M)$

$$g^{\mu\rho} g^{\nu\sigma} (A^{-1})^\xi{}_\rho (A^{-1})^\chi{}_\sigma (h^\dagger) \star A^\alpha{}_\mu A^\beta{}_\nu (h) = g^{\xi\alpha} g^{\chi\beta}.$$

(5)

ρ and hyperbolic ρ Minkowski spacetimes:

$$A(P_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(P_0) & \sin(P_0) & 0 \\ 0 & -\sin(P_0) & \cos(P_0) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(P_3) = \begin{pmatrix} \cosh(P_3) & \sinh(P_3) & 0 & 0 \\ \sinh(P_3) & \cosh(P_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

solve (5) for

$$g(\rho)^{\mu\nu} = \text{diag}(-1, |b|, |b|, 1), \quad b \in \mathbb{R}$$

$$g(H\rho)^{\mu\nu} = \text{diag}(-|b|, |b|, 1, 1), \quad b \in \mathbb{R}$$

Triangular space

$$A(P_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ P_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is solved by Galilean metric $\text{diag}(0, 1, 1, 1)$.