Quantum Foundations III:
Decoherence

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Decoherence in a nutshell:

- In QM relative phase is physical:
$|\psi_1\rangle + e^{i\varphi}|\psi_2\rangle$ not equivalent to $|\psi_1\rangle + e^{i\varphi'}|\psi_2\rangle$

- There are physical mechanisms by which information about the phase is effectively washed out:

$$|\psi_1\rangle + e^{i\varphi}|\psi_2\rangle \to |\psi_1\rangle + e^{i\cdot\mathsf{unknown}}|\psi_2\rangle$$

- Coherent superposition: $e^{i\varphi}$ known.
- Incoherent superposition: $e^{i\varphi}$ unknown.
- Decoherence: transition from coherent → incoherent.

- Today theoretically well understood and experimentally confirmed.
- Nobel prize 2012 (Haroche and Wineland).

The role of decoherence in quantum foundations:

- The mechanism of decoherence does **not depend** on interpretation (Copenhagen, many worlds, Bohmian, ...).
- Yet it significantly **helps** to understand why some quantum systems behave classically.
- For instance, why a quantum cat $|\text{live}\rangle + |\text{dead}\rangle$
appears either $|\text{live}\rangle$ or $|\text{dead}\rangle$.

However:
- Decoherence helps, but it's not enough!
- To explain the origin of classicality **completely**,
it is necessary to use some specific interpretation.

- Here we shall deal mostly with interpretation-independent aspects.

Question: How to write down a state with unknown $e^{i\varphi}$?
Answer: With **density matrix**!

State with known $e^{i\varphi}$:

$$|\psi\rangle = \frac{e^{i\varphi_0}}{\sqrt{2}}\left(|\psi_1\rangle + e^{i\varphi}|\psi_2\rangle\right)$$

Density matrix

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + e^{-i\varphi}|\psi_1\rangle\langle\psi_2| + e^{i\varphi}|\psi_2\rangle\langle\psi_1|\right)$$

- does not depend on the unphysical total phase $e^{i\varphi_0}$
- contains all the physical information as $|\psi\rangle$

In the representation $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\rho \equiv \rho(\varphi) = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\varphi} \\ e^{i\varphi} & 1 \end{pmatrix}$$

Now if $e^{i\varphi}$ is completely unknown, we **average** over all possible values of φ:

$$\tilde{\rho} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \rho(\varphi) = \frac{1}{2}\left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- There is **no** $|\tilde{\psi}\rangle$ such that $\tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}|$.
- State $|\psi\rangle$ or $\rho = |\psi\rangle\langle\psi|$ is called **pure** state.
- State $\tilde{\rho}$ is called **mixed** state.

More generally

$$|\psi\rangle = \sum_k c_k |k\rangle$$

$c_k = |c_k| e^{i\varphi_k}$, average over all phases $\varphi_k \Rightarrow$

$$\tilde{\rho} = \sum_k p_k |k\rangle\langle k| = \begin{pmatrix} p_1 & 0 & \cdots \\ 0 & p_2 & \\ \vdots & & \ddots \end{pmatrix}$$

where $p_k = |c_k|^2$ are probabilities.

$\Rightarrow$ Average over φ_k destroys all non-diagonal elements of ρ in basis $\{|k\rangle\}$.

Destruction of phases equivalent to destruction of interference.

For instance, 2-slit experiment (slits called A and B):

$$\psi(x) = \psi_A(x) + \psi_B(x)$$

$$\psi_A(x) = e^{i\varphi_A(x)}|\psi_A(x)|, \quad \psi_B(x) = e^{i\varphi_B(x)}|\psi_B(x)|$$

Probability density in the position space $p(x) = |\psi(x)|^2 \Rightarrow$

$$p(x) = \underbrace{|\psi_A(x)|^2 + |\psi_B(x)|^2}_{\text{classical sum of probabilities}} + \underbrace{2\cos\left[\varphi_A(x) - \varphi_B(x)\right]|\psi_A(x)||\psi_B(x)|}_{\text{quantum interference}}$$

Average over phases $\Rightarrow$

$$\tilde{p}(x) = |\psi_A(x)|^2 + |\psi_B(x)|^2$$

$\Rightarrow$ **Decoherence implies that quantum addition of probability amplitudes gets replaced by classical addition of probabilities.**

- Important step towards understanding how classical physics emerges from fundamental quantum laws.

However, we still haven't answered the most important question:
Why is the phase unknown?

There can be various reasons, but the most common reason is:
Due to interaction with environment!
- This is called **environment-induced decoherence**.
- When physicists talk about "decoherence", they usually mean **this** kind of decoherence.
⇒ In the rest we study only environment-induced decoherence.

- Other types of "decoherence" better known under the name **dephasing**.

Heuristic explanation:

$|\psi\rangle$ a state in the system we want to study

$|\phi\rangle$ a state in the environment

$\Rightarrow$ A state in the full system is a superposition of terms like

$$(e^{i\varphi}|\psi\rangle)|\phi\rangle = |\psi\rangle(e^{i\varphi}|\phi\rangle) \equiv |\psi\rangle|\phi'\rangle$$

$\Rightarrow$ The phase is "eaten up" by environment.

$\Rightarrow$ If we don't know the state of environment,
the effect is the same as if we don't know the phase.

Why do we not know the state of environment?

- Sometimes simply because we choose not to measure it.
- More often, because it has large (say 10^{23}) number of degrees of freedom, so it's **impossible** to measure it in practice.

In the rest we make these heuristic ideas more precise.

Average $=$ trace:

For pure state $|\psi\rangle$

$$\begin{aligned}\langle A\rangle &= \langle\psi|A|\psi\rangle = \sum_k\sum_{k'}\langle\psi|k'\rangle\langle k'|A|k\rangle\langle k|\psi\rangle = \sum_k\sum_{k'}\underbrace{\langle k\overbrace{|\psi\rangle\langle\psi|}^{\rho}k'\rangle}_{\rho_{kk'}}\underbrace{\langle k'|A|k\rangle}_{A_{k'k}} \\ &= \sum_k(\rho A)_{kk} = \mathrm{Tr}\rho A = \mathrm{Tr}A\rho\end{aligned}$$

Probability: $p_k = |\langle k|\psi\rangle|^2 = \langle\psi\overbrace{|k\rangle\langle k|}^{\pi_k}\psi\rangle = \langle\pi_k\rangle$, π_k is projector.

For two subsystems with bases $\{|k\rangle\}$ and $\{|l\rangle\}$ the full basis is $\{|k\rangle|l\rangle\}$.

$$\mathrm{Tr}O = \sum_k\sum_l\langle l|\langle k|O|k\rangle|l\rangle = \mathrm{Tr}_2\mathrm{Tr}_1O = \mathrm{Tr}_1\mathrm{Tr}_2O$$

where $\mathrm{Tr}_1O = \sum_k\langle k|O|k\rangle$ and $\mathrm{Tr}_2O = \sum_l\langle l|O|l\rangle$ are called **partial traces**.

For operator in the first subsystem $A = A_1 \otimes 1$

$$\mathrm{Tr} A\rho = \mathrm{Tr}_1 \mathrm{Tr}_2 A\rho = \mathrm{Tr}_1 (A_1 \mathrm{Tr}_2 \rho) = \mathrm{Tr}_1 A_1 \rho_1$$

where

$$\boxed{\rho_1 = \mathrm{Tr}_2 \rho}$$

is called **reduced density matrix**.

$\Rightarrow$ **All quantum information about the first subsystem is encoded in** $\rho_1 = \mathrm{Tr}_2 \rho$**.**

Example:

$$|\Psi\rangle = \sum_k c_k |k\rangle |\phi_k\rangle$$

The first subsystem is **not** $|\psi\rangle = \sum_k c_k |k\rangle$. Instead, if $\langle\phi_k|\phi_{k'}\rangle = \delta_{kk'}$

$$\rho_1 = \mathrm{Tr}_2 |\Psi\rangle\langle\Psi| = \sum_k p_k |k\rangle\langle k| \neq |\psi\rangle\langle\psi|, \quad p_k = |c_k|^2$$

$\Rightarrow$ ρ_1 is a mixed state, the information about phases of c_k is “lost”!

- More precisely, in principle information is still there in the full $|\Psi\rangle$, but it's not visible in the first subsystem alone.

- Similarly, it's not visible in the second subsystem alone:

$$\rho_2 = \mathrm{Tr}_1 |\Psi\rangle\langle\Psi| = \sum_k p_k |\phi_k\rangle\langle\phi_k|$$

Partial decoherence:

What if $\langle\phi_k|\phi_{k'}\rangle \neq \delta_{kk'}$?
Example:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle|\phi_1\rangle + |2\rangle|\phi_2\rangle)$$

$\rho_1 = \mathrm{Tr}_2|\Psi\rangle\langle\Psi| \Rightarrow$

$$\begin{aligned}\rho_1 &= \tfrac{1}{2}(|1\rangle\langle 1| + |2\rangle\langle 2| + |1\rangle\langle 2|\langle\phi_2|\phi_1\rangle + |2\rangle\langle 1|\langle\phi_1|\phi_2\rangle) \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & \langle\phi_2|\phi_1\rangle \\ \langle\phi_1|\phi_2\rangle & 1 \end{pmatrix}\end{aligned}$$

Full decoherence (full diagonalisation): $\langle\phi_1|\phi_2\rangle = 0$
Full coherence: $|\langle\phi_1|\phi_2\rangle| = 1$
Partial decoherence: $0 < |\langle\phi_1|\phi_2\rangle| < 1$

Full coherence $\Rightarrow$ no entanglement:

$$|\langle\phi_1|\phi_2\rangle| = 1 \;\Rightarrow\; |\phi_2\rangle = e^{i\varphi}|\phi_1\rangle$$

$$\Rightarrow\; |\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + e^{i\varphi}|2\rangle)|\phi_1\rangle \equiv |\psi\rangle|\phi_1\rangle$$

- When the second subsystem has **many** degrees of freedom then decoherence is usually almost full: $|\langle\phi_1|\phi_2\rangle| \ll 1$.

Simple explanation:
- For $N \gg 1$ degrees, usually they are statistically quite independent.
- Independent means not correlated,
which in QM means not entangled: $|\phi_k\rangle \simeq |\chi_{k1}\rangle \cdots |\chi_{kN}\rangle$

$$\Rightarrow \ \langle\phi_1|\phi_2\rangle \simeq \langle\chi_{11}|\chi_{21}\rangle \cdots \langle\chi_{1N}|\chi_{2N}\rangle, \quad |\langle\chi_{1j}|\chi_{2j}\rangle| \equiv e^{-\alpha_j} < 1$$

For simplicity we can take all α_j to be roughly the same $\alpha_j \sim \alpha$

$$\Rightarrow \ |\langle\phi_1|\phi_2\rangle| \sim e^{-\alpha N} \ll 1$$

- The second system with many degrees usually called **environment**.
- A particularly useful kind of environment: **macroscopic measuring apparatus**.

Quantum measurement:

- Suppose that $|\phi_1\rangle$ and $|\phi_2\rangle$ are two macroscopic states of the measuring apparatus.
- In a practical sense, the apparatus "measures" only if the two states can be **distinguished** $\Rightarrow |\langle\phi_1|\phi_2\rangle| \ll 1$.

$\Rightarrow$ **Measurement is associated with (almost) full decoherence.**

Generalized measurement with n possible outcomes:

$$|\Psi\rangle = \sum_{l=1}^{n} c_l |\psi_l\rangle |\phi_l\rangle, \quad \langle\phi_l|\phi_{l'}\rangle \simeq \delta_{ll'}$$

Counter-example:

- In quantum optics, a mirror does not destroy coherence of a photon.

$\Rightarrow$ Mirror does not cause decoherence.

$\Rightarrow$ Mirror does not measure the photon. Why?

$|\phi_1\rangle$ - state of mirror when it is not hit by the photon

$|\phi_2\rangle$ - state of mirror when it is hit by the photon

- When photon hits the mirror, it transfers momentum $\mathbf{p}$ to the mirror.

$x =$ position of the mirror center of mass in the direction of $\mathbf{p}$

$... =$ all the other mirror particle positions

- Mirror is a correlated system of particles, all momentum transferred to x

$$\Rightarrow \; \phi_2(x,...) = e^{ipx/\hbar}\phi_1(x,...) \; \Rightarrow \; \langle\phi_1|\phi_2\rangle \propto \int dx\, e^{ipx/\hbar}|\phi_1(x,...)|^2$$

- Center of mass is well defined, i.e. $\phi_1(x,...)$ is narrow Gaussian in x.

$\Rightarrow$ $e^{ipx/\hbar}$ nearly a constant inside the Gaussian

(unless the transferred momentum p is very big)

$$\Rightarrow \; |\langle\phi_1|\phi_2\rangle| \simeq 1$$

i.e. the photon is not much decohered by mirror.

Dynamics of decoherence:

- Evolution of the full closed system is unitary: $|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle$
- Due to interactions, evolution of a subsystem is usually not unitary.

$\Rightarrow$ $|\langle\phi_1(t)|\phi_2(t)\rangle| \equiv r(t)$ (decoherence factor) changes with time.

Heuristic description:
- Initially (before interaction) $r(0) = 1$.
- During a short time δt it slightly lowers

$$r(\delta t) = r(0) - \Gamma\delta t = r(0)[1 - \Gamma\delta t] \simeq r(0)e^{-\Gamma\delta t}$$

- Due to many degrees of freedom, the system soon "forgets" its initial state, so it always repeats the same decay pattern:

$$r(2\delta t) \simeq r(\delta t)e^{-\Gamma\delta t} \simeq r(0)e^{-\Gamma 2\delta t}$$

$\Rightarrow$ For long times exponential decay

$$r(t) \simeq r(0)e^{-\Gamma t}$$

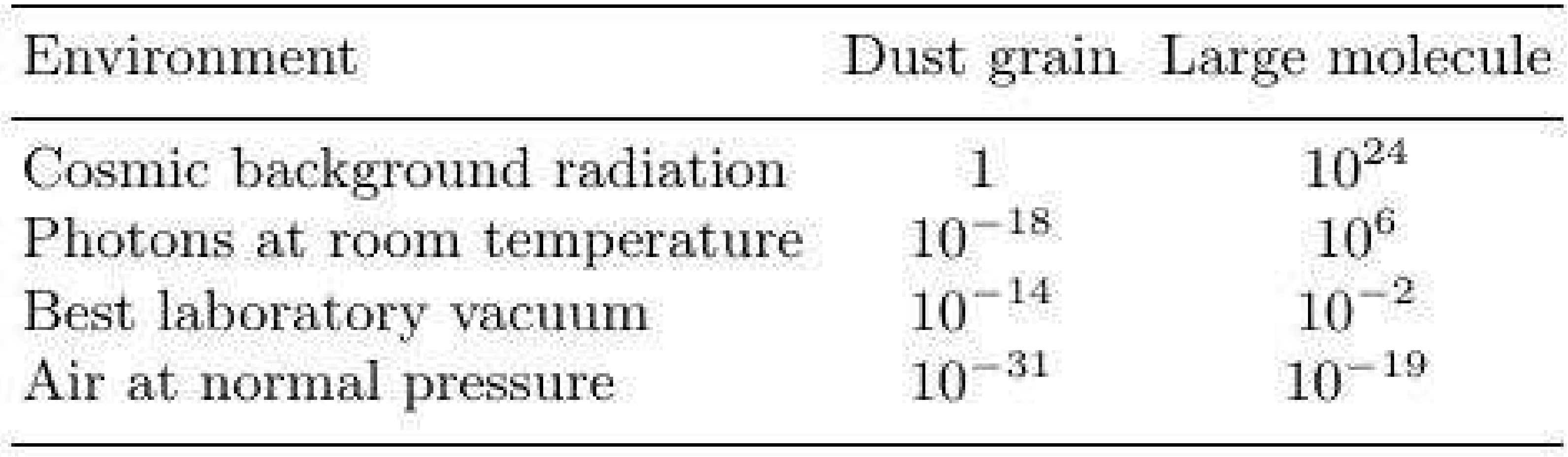
- More careful computations and simulations qualitatively confirm.

- At extremely long times ($t \gg$ age of the universe) $r(t)$ can grow (quantum Poincare recurrence).
- Irrelevant in practice.
- For all practical purposes (FAPP), decoherence is irreversible.
- It's a statistical law, closely related to 2nd law of thermodynamics.

Γ depends on details of the interaction Hamiltonian.
- For N degrees of freedom, usually $\Gamma \propto N$.

$\Rightarrow$ For macroscopic objects decoherence is usually **very fast**.

Typical decoherence times (in seconds) $\tau = 1/\Gamma$ for some cases:



Environment	Dust grain	Large molecule
Cosmic background radiation	1	10^{24}
Photons at room temperature	10^{-18}	10^{6}
Best laboratory vacuum	10^{-14}	10^{-2}
Air at normal pressure	10^{-31}	10^{-19}

Environment-induced superselection:

- Decoherence: mechanism by which ρ_1 becomes (nearly) diagonal.
- But if ρ_1 is diagonal in one basis, it's not diagonal in another.

Question: What determines the basis in which ρ_1 is diagonal?
Answer: The Hamiltonian (of the full closed system)!

- If $H|\Psi(t)\rangle \neq E|\Psi(t)\rangle \Rightarrow |\Psi(t)\rangle$ is non-stationary, changes with time.
- If $H|\Psi(t)\rangle = E|\Psi(t)\rangle \Rightarrow |\Psi(t)\rangle = e^{-iEt/\hbar}|\Psi(0)\rangle$ does not change (the overall phase $e^{-iEt/\hbar}$ is irrelevant).

$\Rightarrow$ H-eigenstates are stable, other states are unstable.
$\Rightarrow$ Decoherence factors $r(t) \simeq e^{-\Gamma t}$ decay in the basis of H-eigenstates.
$\Rightarrow$ After time $t \sim \Gamma^{-1}$, ρ_1 settles down into a matrix (nearly) diagonal in the basis of H-eigenstates.

$\Rightarrow$ H defines a preferred basis.
- In this basis, coherent superpositions are often (FAPP) impossible.
- In general, impossibility of a certain kind of coherent superposition is called **superselection**.
- Decoherence explains superselection via interaction with environment.

Examples:

Charge Q:

- Exactly conserved $\Rightarrow [Q, H] = 0$.

$\Rightarrow$ Impossible to find a coherent superposition of states with different charges, e.g. $|1 \text{ electron}\rangle + |2 \text{ electrons}\rangle$

Energy of the 1st subsystem:

$H = H_1 + H_2 + H_{\text{int}}$

- If $H_{\text{int}} \ll H_1 \Rightarrow H$-eigenstates are approximately proportional to H_1-eigenstates.

$\Rightarrow \rho_1$ is nearly diagonal in the basis of H_1-eigenstates (e.g. a stable or quasi-stable atom).

Interaction energy:

- If $H_{\mathrm{int}} \gg H_1 + H_2 \Rightarrow H$-eigenstates are close to H_{int}-eigenstates.

$\Rightarrow \rho_1$ is nearly diagonal in the basis of H_{int}-eigenstates.

- Usually H_{int} depends only on positions, e.g. $H_{\mathrm{int}} = V(r)$.
- Explains why do we not observe macroscopic superpositions of states at different places, e.g. $|\text{cat here}\rangle + |\text{cat there}\rangle$.

Application to Schrödinger cat:

- Difference between life and death can be reduced to a difference between positions of some macroscopic objects

(e.g. eyelid open *vs* eyelid closed).

$\Rightarrow$ Decoherence explains why there are **no** coherent superpositions $|\text{cat live}\rangle + |\text{cat dead}\rangle$.

Does decoherence explain the wave-function collapse?

- Not quite!
- Decoherence eliminates certain **coherent** superpositions,
but it still contains **incoherent** superpositions.

For coherent superposition $|\psi\rangle = \frac{1}{\sqrt{2}}(|\text{live}\rangle + |\text{dead}\rangle)$
in the representation $|\text{live}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\text{dead}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have

$$|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{decoherence}} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

On the other hand, collapse corresponds to an additional transition:

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{collapse}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{collapse}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Mixed state ρ always related to a lack of knowledge.
- Physical reasons for the lack of knowledge may be different, but ρ may look the same.

Improper mixture:
- Full system $|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle|\phi_1\rangle + |2\rangle|\phi_2\rangle)$
- Someone who doesn't measure the 2nd subsystem, describes the 1st subsystem as $\rho_1 = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2|$

Proper mixture:
- An apparatus prepares a pure state, either $|1\rangle$ or $|2\rangle$.
- The decision is made by a classical pseudo-random mechanism.
- Someone who doesn't know what the decision was, describes her knowledge as $\rho = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2|$

- For given ρ, the interpretation as a proper mixture is not unique.

E.g. $|1\rangle\langle 1| + |2\rangle\langle 2| = |+\rangle\langle +| + |-\rangle\langle -|$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$

The problem of measurement in QM can be expressed as:
- How does an improper mixture turn into a proper mixture?
- The answer depends on the interpretation of QM (Copenhagen, many worlds, Bohmian, ...)

$\Rightarrow$ Decoherence **alone** doesn't solve the measurement problem!

Nevertheless, decoherence has far reaching implications:
- Coherent superposition of a macroscopic object
(e.g. Schrödinger cat) can live only for a very short time.
- Once coherence gets destroyed, it remains destroyed “forever” (FAPP).
- It’s very hard to build a quantum computer with many qbits.

Relevance for solution of the measurement problem:
- Fast decoherence **helps** to explain why macro objects obey classical laws.
- Irreversibility of decoherence **helps** to explain the (illusion of) wave-function collapse.
- A **full** solution requires a use of some specific interpretation
(Copenhagen, many worlds, Bohmian, ...)
not to be discussed today.

Forthcoming talks:

Quantum Foundations IV:
Quantum theory of measurement and Bohmian mechanics

Quantum Foundations V:
Relativistic QFT from a Bohmian perspective: A proof of concept

Quantum Foundations VI:
Suggestions welcome