Gauge theories on noncommutative spacetimes

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In this seminar we review the current status of development of gauge theories on noncommutative spacetimes. We mostly follow the recent review paper [1] and expand it with author's personal contributions to the field.

I. INTRODUCTION

From a modern mathematical perspective, noncommutative geometry is understood as the study of noncommutative algebras, the modules defined over them, and the geometric structures that can be reconstructed from such algebraic data. This viewpoint reflects a broader philosophy in which geometric spaces are replaced by suitable algebras of functions, and geometric notions are reformulated in purely algebraic terms. The foundation of this approach is provided by several fundamental results, most notably:

- The Gelfand–Naimark theorem, which establishes that every commutative C^*-algebra is isometrically isomorphic to the algebra of continuous complex-valued functions on a compact Hausdorff space [2].
- The Serre–Swan theorem, which gives a one-to-one correspondence between finitely generated projective modules over a commutative algebra and vector bundles over the corresponding space [3, 4].
- Connes' reconstruction theorem, which shows that every commutative spectral triple arises from a compact spin manifold, thereby providing an algebraic analogue of Riemannian geometry [5].

Quantum gauge field theory, celebrated for its central role in the formulation of the Standard Model of particle physics, provides a natural arena in which to apply and test the ideas of noncommutative geometry. At sufficiently high energies, it is widely expected that the classical description of spacetime as a smooth manifold ceases to be adequate and must be replaced by a noncommutative structure [6]. In such regimes, the coordinate functions of spacetime are anticipated to become noncommuting observables, leading to uncertainty relations among spacetime coordinates [6]. Similar phenomena appear in certain limits of string theory, where noncommutative geometry arises naturally in the presence of background fields [7]. One motivation for this expectation is that coordinate uncertainty may offer a resolution to the paradox in which arbitrarily precise localization of events would require energies large enough to form a black hole, thereby obstructing the measurement itself [6].

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For these reasons, the formulation of quantum gauge theories on noncommutative spaces is of interest both as a possible avenue toward physics beyond the Standard Model and as a testing ground for the applicability of noncommutative geometry to fundamental physical problems.

In this seminar, we will review the main approaches that have been developed to define and study gauge theories on noncommutative spaces, and discuss their conceptual and technical challenges.

The seminar is organized as follows: in Section II we introduce the convolution algebra quantization which can realize all of the noncommutative spacetimes considered in this seminar. We also introduce the concepts of connection and curvature on modules of noncommutative algebras as well as their gauge transformations. In Section III, IV and V, we review the current progress of gauge theories on Moyal $\mathbb{R}^{2n}_\theta$, $\mathbb{R}^3_\lambda$ and κ–Minkowski $\mathbb{R}^4_\kappa$ spacetimes respectively. We finish off with Section VI with concluding remarks and where we present author's soon to be published result of a unified construction of $U(N)$ gauge theory on almost-abelian unimodular noncommutative spacetimes, which includes the well studied ρ–Minkowski spacetime and various new, completely unexplored quantum spaces.

II. NC GEOMETRY PRELIMINARIES

As anticipated, in this section we introduct the necessary NC geometry preliminaries neccesary to understand the rest of this seminar. We start by explaining a procedure to deform the pointwise algebras of complex functions on the manifold.

II.1. Weyl-like convolutional quantization

Consider the algebra of smooth complex functions on a manifold, e.g., the $*$-algebra $C^\infty(M,\mathbb{C})$. In order to promote this commutative algebra into a noncommutative algebra with prescribed commutators between coordinates, one can define the noncommutative product, so called $\star$ product, which realizes the prescribed coordinate commutators.

The procedure is the following: impose that the coordinate functions are generators of a Lie algebra $\mathfrak{g}$ and then consider Fourier transforms of complex spacetime functions

$$\mathbb{C} \ni \tilde{f}(p) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n x \, f(x) e^{ipx} \, . \tag{1}$$

The Fourier transforms are now complex functions on the Lie group $\mathcal{G}$ (corresponding to $\mathfrak{g}$) and p is a coordinate chart on the group. Then, one can trivially add such functions, but also multiply them using the group convolution algebra machinery

$$(\tilde{f} \circ \tilde{g})(p) = \int_{\mathcal{G}} \tilde{f}(pq)\tilde{g}(q^{-1}) \, d\mu^L_{\mathcal{G}}(q) \, , \tag{2}$$

where $d\mu^L_{\mathcal{G}}$ is the left-Haar measure on the Lie group $\mathcal{G}$ and pq corresponds to the coordinates of the group element on $\mathcal{G}$ given as the product of group elements with coordinates p and q. Similarly, q^{-1} are the coordinates of the group element that is inverse to the group element with coordinates q. The convolution algebra of any Lie group $\mathcal{G}$ is a $*$-algebra with the involution given as

$$\tilde{f}^{\dagger}(p) = \tilde{f}^{*}(p^{-1}) \Delta_{\mathcal{G}}(p^{-1}) \tag{3}$$

with $\Delta_{\mathcal{G}}$ the modular function on the Lie group. Finally, the $\star$ product of two functions can be defined as the inverse Fourier transform of the convolution product

$$(f \star g)(x) = \mathcal{F}^{-1}(\tilde{f} \circ \tilde{g}) \, . \tag{4}$$

From the properties of Fourier transforms and convolutional product it follows that the algebra $C^\infty_\star(M) = (C^\infty(M), \star)$ is a unital noncommutative $*$-algebra with elements given as complex spacetime functions.

Concretely, this construction always works if $\mathcal{G}$ is locally compact [1]. This construction is based on [8–12].

II.2. Connections and curvature on modules over noncommutative algebras

The second key ingredient in constructing gauge-invariant action functionals from noncommutative geometric data is a suitable notion of connection and curvature, together with their gauge transformations, formulated purely algebraically [13].

In ordinary gauge theory, the gauge potential is often introduced as a Lie algebra–valued one-form on a principal bundle $P \to M$. Geometrically, this one-form encodes a connection on the principal bundle which defines a splitting of the tangent space at each point of P into a vertical and horizontal subspace. The horizontal subspaces specify how to lift paths on the base manifold M to the bundle, and the Lie algebra–valued one-form assigns infinitesimal parallel transport along these directions. On the other hand, even in the commutative setting, connections admit a completely algebraic formulation.

Let $E \to M$ be a vector bundle over a smooth manifold M, with $\Gamma(E)$ its module of sections. For any vector field $X \in \Gamma(TM)$, a directional covariant derivative along X is a map

$$\nabla_X : \Gamma(E) \longrightarrow \Gamma(E) \tag{5}$$

satisfying the Leibniz rule

$$\begin{aligned} &\nabla_X(fs) = X(f)\, s + f\, \nabla_X(s) \, , \\ &f \in C^\infty(M), \quad s \in \Gamma(E), \end{aligned} \tag{6}$$

where $X(f)$ is the action of the derivation X on the function $f \in C^\infty(M)$. Equivalently, the covariant derivative can be expressed as a map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(E) \otimes_{C^\infty(M)} \Omega^1(M), \tag{7}$$

related to the directional derivatives via

$$\nabla_X(s) = \nabla(s)(1 \otimes X) \, , \tag{8}$$

where we view X as a derivation of the algebra $C^\infty(M)$. This algebraic formulation depends only on the algebra $C^\infty(M)$, its module of sections $\Gamma(E)$, and the differential calculus $\Omega^1(M)$, making it naturally generalizable to the noncommutative setting: one replaces $C^\infty(M)$ with a noncommutative algebra $\mathcal{A}$, $\Gamma(E)$ with an $\mathcal{A}$–module $\mathcal{E}$, and derivations X with elements of $\mathrm{Der}(\mathcal{A})$.

A complication which should not be dismissed is the fact that in noncommutative geometry commutators with coordinates are derivations

$$[f, a \star b]_\star = f \star a \star b - a \star b \star f = [a, f]_\star \star b + a \star [f, b]_\star \, , \tag{9}$$

which will tend to the 0 derivation in any kind of commutative limit. With that said, the space $\mathrm{Der}(\mathcal{A})$ is very big for an arbitrary noncommutative algebra, so usually we will work with a finitely generated subspace $\mathcal{D} \leq \mathrm{Der}(\mathcal{A})$ of the full space of derivations.

As a concrete example, for $U(N)$ gauge theory on flat Minkowski spacetime $\mathcal{M}$, the associated vector bundle is trivial:

$$E \cong \mathcal{M} \times \mathbb{C}^N, \quad \Gamma(E) \cong C^\infty(\mathcal{M})^{\oplus N}. \tag{10}$$

Algebraically, this corresponds to the free module

$$\mathcal{E} = \mathcal{A}^{\oplus N}, \qquad \mathcal{A} = C^\infty(\mathcal{M}), \tag{11}$$

and the directional covariant derivatives ∇_X encode the action of the Lie algebra–valued 1–form, the gauge potential, on the derivations X. This motivates studying connections of modules of the form $\mathcal{E} = \mathcal{A}^{\oplus N}$ in the noncommutative setting.

Having said all of this, let us take a step back and consider a general associative (possibly noncommutative) algebra $\mathcal{A}$ with the product that we denote as $\star$. Let $\mathcal{D} \leq \mathrm{Der}(\mathcal{A})$ be a finitely generated subspace of derivations on $\mathcal{A}$ and let E be a right $\mathcal{A}$–module. In other words, E is a complex vector space endowed with a right $\mathcal{A}$ action $\triangleleft$ that respects the algebra structure of $\mathcal{A}$ (being a right action, it is a homomorphism $\mathcal{A}^{\mathrm{op}} \to \mathrm{Hom}(E, E)$).

With this setup, a right connection on the right module E is defined as the map $\nabla_X : E \to E$ (with $X \in \mathcal{D}$) satisfying

$$\begin{aligned} \nabla_X(m \lhd a) &= m \lhd X(a) + \nabla_X(m) \lhd a \ , \\ \nabla_{b\star X+Y}(m) &= \nabla_X(m) \lhd b + \nabla_Y(m) \ , \end{aligned} \tag{12}$$

for all $a \in \mathcal{A}, b \in \mathcal{Z}(\mathcal{A}), m \in E$ where $\mathcal{Z}(\mathcal{A})$ is the center of $\mathcal{A}$ (i.e., the set of commuting elements in $\mathcal{A}$). Using the connection ∇_X, we can also define the curvature $F(X, Y)$ for $X, Y \in \mathcal{D}$ as the right $\mathcal{A}$-linear map

$$F(X, Y)(m) = [\nabla_X, \nabla_Y](m) - \nabla_{[X,Y]}(m) \ . \tag{13}$$

II.3. Gauge transformations

Continuing in the algebraic setup [13] of considering an algebra $\mathcal{A}$ along with its derivations $\mathrm{Der}(\mathcal{A})$ and a right–$\mathcal{A}$ module E, in this subsection we will give results which generalize gauge transformations of connections and curvatures to this general noncommutative scenario.

The results are quite simple. Namely, suppose that $\phi : E \to E$ is a right $\mathcal{A}$ automorphism of the module E, i.e., ϕ is a linear map on E, is bijective and it is compatible with the module structure

$$\phi(m \lhd a) = \phi(m) \lhd a. \tag{14}$$

hence, provided that ∇_X is a connection on the module E in the sense as explained in the prior subsection, then ∇^ϕ_X defined as

$$\nabla^\phi_X = \phi^{-1} \circ \nabla_X \circ \phi \tag{15}$$

is also a connection. Additionally, the curvature of ∇^ϕ_X, $F^\phi(X, Y)$, is related to the curvature of ∇_X, $F(X, Y)$, as follows

$$F^\phi(X, Y) = \phi^{-1} \circ F(X, Y) \circ \phi. \tag{16}$$

II.4. Example - $U(1)$ gauge structure on noncommutative modules

Now, given an algebra $\mathcal{A}$ and its derivations $\mathrm{Der}(\mathcal{A})$, consider the right $\mathcal{A}$-module as $\mathcal{A}$ itself, i.e., $E = \mathcal{A}$. Additionally, suppose that $\mathcal{A}$ is a unital algebra, otherwise automorphisms on $E = \mathcal{A}$, which are required for the definition of gauge transformations, do not make sense. The automorphisms on $\mathcal{A}$ are $\star$-multiplications by invertible elements, i.e., for an invertible $g \in \mathcal{A}$,

$$\phi_g(a) := a \star g \ . \tag{17}$$

The results (15) and (16) are immediately applicable, provided a connection on $\mathcal{A}$ exists. From the Leibniz rule, selecting $m = 1$, the unit element of $\mathcal{A}$, we find

$$\nabla_X(1 \lhd a) = 1 \lhd X(a) + \nabla_X(1) \lhd a \ . \tag{18}$$

In other words,

$$\nabla_X(a) = X(a) + \nabla_X(1) \star a \ , \tag{19}$$

so a connection ∇_X is uniquely determined by its action on the unit element. Following the standard physics notation, we can rename $\nabla_X(1) = iA_X$ so we can see that $-i\nabla(1)$ plays the role of the gauge potential 1–form A, acting on derivations (vector fields) to produce algebra elements (functions). The same calculation can be performed on the gauge transformed connection ∇^ϕ_X and one finds the relation

$$A^\phi_X \equiv \nabla^\phi_X(1) = g^{-1} \star A_X \star g - ig^{-1} \star X(g) \ , \tag{20}$$

which resembles the noncommutative version of the well known $U(1)$ gauge transformation rule. To make the link with electrodynamics complete, one needs to consider only the automorphisms on $\mathcal{A}$ which obey a unitarity relation. Thus, if $\mathcal{A}$ is an involutive algebra with the involution †, one can consider the subgroup of automorphisms given as

$$\mathcal{U}(\mathcal{A}) = \left\{ g \in \mathcal{A} : g \star g^\dagger = g^\dagger \star g = 1 \right\} \leq \mathrm{Aut}(\mathcal{A}) \ , \tag{21}$$

which when applied to (20) gives

$$A^g_X \equiv \nabla^g_X(1) = g^\dagger \star A_X \star g - ig^\dagger \star X(g) \ . \tag{22}$$

It is worth mentioning that all of the results here hold analogously for left-modules and left-connections. Additionally, when $E = \mathcal{A}^{\oplus N}$ (the $U(N)$ case), the gauge group $\mathcal{U}(E)$ is given as

$$\mathcal{U}(E) = \left\{ g_{ij} \in \mathrm{Mat}_{N\times N}(\mathcal{A}) : g_{ik} \star g^\dagger_{kj} = \delta_{ij} \right\} \ . \tag{23}$$

Finally, other gauge groups, like $SU(N)$, are largely unexplored in noncommutative geometry due to the inherent difficulty of defining the notion of noncommutative determinants. In commutative algebra, many properties of matrix determinants rely on the commutativity of the product in order to simplify to the known theorems (e.g. Binet-Cauchy theorem etc.). In noncommutative geometry the product is, of course, noncommutative so many properties of the determinant do not hold.

III. GAUGE THEORY ON MOYAL SPACES $\mathbb{R}^{2n}_\theta$

In this section we review the Moyal spacetime and the gauge theory constructions on it.

III.1. Moyal product

Moyal spacetime is defined by the following commutation relation of coordinates

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu} \ . \tag{24}$$

This commutation relation can be realized with a star product obtained in the way described in Section II.1. We shall demonstrate this now, but, to simplify consider first a space of two coordinates, x^1 and x^2 and impose the noncommutativity

$$\left[x^1, x^2\right]_\star = i\theta \; . \tag{25}$$

This commutation relation is not of a Lie algebraic type (θ is not a generator of coordinate algebra), so at first the procedure from Section II.1 is not applicable. But, one can first pretend that θ is also a generator of coordinate algebra, promoting (25) to a commutation relation for the Lie algebra $\mathfrak{h}_3$ of the 3D Heisenberg Lie group H_3, and then later do a type of contraction to remove the coordinate dependence of functions on θ. The group laws of the Heisenberg group are well known

$$\begin{aligned} g(z,u,v)g(p,q,r) &= g(z+p+\frac{1}{2}\left(uq-pv\right), u+q, v+r) \\ g^{-1}(z,u,v) &= g(-z,-u,-v) \end{aligned} \tag{26}$$

and this immediately defines a convolution algebra structure on complex group functions $F(z,u,v) \in C^\infty(H_3, \mathbb{C})$. A technical trick, used for reducing to two variables, is to now introduce a map $^\# : L^1(\mathbb{R}^3) \to L^1(\mathbb{R}^2)$ defined as

$$F^\#(u,v) := \int_{\mathbb{R}} F(z,u,v) e^{-i2\pi\theta z} \tag{27}$$

which respects a twisted convolution algebra property

$$(F \circ G)^\# = F^\# \;\hat{\circ}\; G^\# \tag{28}$$

with

$$\begin{aligned} &\left(F^\# \;\hat{\circ}\; G^\#\right)(u,v) = \\ &\int_{\mathbb{R}^2} dp\, dq\; F^\#(p,q)\; G^\#(u-p, v-q) e^{i\pi\theta(up-vq)} \; . \end{aligned} \tag{29}$$

Finally, we can use (29) to define a star product, denoted as $\star_\theta$, as follows:

$$(f \star_\theta g)(x^1, x^2) = \mathcal{F}^{-1}\left(\mathcal{F}f \;\hat{\circ}\; \mathcal{F}g\right) \; , \tag{30}$$

which has a closed form given as

$$(f \star_\theta g) = \frac{1}{\pi\theta} \int d^2y\, d^2z\; f(x+y) f(x+z) e^{-2i\theta y^T J z} \tag{31}$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; . \tag{32}$$

It can be shown [1] that, as a formal expansion in θ, the integral (31) is equivalent to the Moyal-Groenewold product

$$f \star_\theta g = \cdot \left[e^{\frac{i}{2}\theta \partial_1 \wedge \partial_2} f \otimes g\right] \tag{33}$$

which appears in the Hopf algebraic approach to noncommutative geometry. The product (31) (or equivalently, (33)) reproduces the commutation relation (25). The described procedure can be generalized from the toy model (25) to the general form (24) in a similar way. The result in the differential form is the following $\star_\theta$ product

$$f \star_\theta g = \cdot \left[e^{\frac{i}{2}\theta^{\mu\nu} \partial_\mu \otimes \partial_\nu} f \otimes g\right] \tag{34}$$

III.2. Gauge structure

The gauge objects from Section II.2 can be defined from the following ingredients:

- Set the algebra $\mathcal{A} = (C^\infty(\mathbb{R}^{3,1}, \mathbb{C}), \star)$
- Choose its right module $E = \mathcal{A}^{\oplus N}$
- Choose the finitely generated subspace of $\mathrm{Der}(\mathcal{A})$, $\mathcal{D}$, given as $\mathcal{D} = \mathrm{span}_{\mathcal{Z}(\mathcal{A})}\left\{\partial_\mu : \mu = 0, ..., 3\right\}$ where $\mathcal{Z}(\mathcal{A})$ is the center of the algebra $\mathcal{A}$.

The reason for this choice of $\mathcal{D}$ is for it to mimic the module of sections of the tangent bundle, which is finitely generated with partial derivatives. With this data, the $U(N)$ Yang-Mills field is given as

$$A_\mu = -i \nabla_{\partial_\mu}(1) \tag{35}$$

and the field-strength tensor is given as

$$F_{\mu\nu} = F(\partial_\mu, \partial_\nu)(1) = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right]_\star \; . \tag{36}$$

III.3. Yang Mills theory on Moyal spacetime

The following action principle, studied in [14–18], is a natural analog of the $U(N)$ action functional on the noncommutative Moyal spacetime

$$S = \int d^4\, x\; \mathrm{Tr}(F_{\mu\nu} \star F_{\rho\sigma}) \eta^{\mu\rho} \eta^{\nu\sigma} \tag{37}$$

and it is gauge invariant. Namely, we have seen that the curvature transforms as

$$F^g_{\mu\nu} = g \star F_{\mu\nu} \star g^\dagger \tag{38}$$

so the gauge transformation of the action gives

$$\begin{aligned} S^g &= \int d^d\, x\; \mathrm{Tr}(g \star F_{\mu\nu} \star_\theta F_{\rho\sigma} \star g^\dagger) \eta^{\mu\rho} \eta^{\nu\sigma} \\ &= \int d^d\, x\; g_{ai} \star_\theta F^{ik}_{\mu\nu} \star_\theta F^{kj}_{\rho\sigma} \star_\theta g^\dagger_{ja} \eta^{\mu\rho} \eta^{\nu\sigma} \end{aligned} \tag{39}$$

But, the integral is cyclic with respect to the Moyal product

$$\int d^{2n}x\; f \star_\theta g = \int d^{2n}x\; g \star_\theta f = \int d^{2n}x\; fg \tag{40}$$

so in the end we indeed do achieve gauge invariance. It is worth to comment that the Minkowski metric η is central in the deformed algebra and the star product with it reduces simply to the pointwise product complex product.

The action (37), due to the commutator term in (36), produces cubic and quartic vertices whose expressions in the momentum space are given as

$$\begin{aligned} &C^3_{\alpha\beta\gamma}(k_1,k_2,k_3) = \\ &-2i\sin\left(\frac{k_1\wedge k_2}{2}\right)\cdot\Big[(k_2-k_1)_\gamma\eta_{\alpha\beta}+ \\ &(k_1-k_3)_\beta\eta_{\alpha\gamma}+(k_3-k_2)_\alpha\eta_{\beta\gamma}\Big] \end{aligned} \tag{41}$$

and

$$\begin{aligned} &V^4_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = -4\cdot \\ &\Big[(\eta_{\alpha\gamma}\eta_{\beta\delta}-\eta_{\alpha\delta}\eta_{\beta\gamma})\sin\left(\frac{k_1\wedge k_2}{2}\right)\sin\left(\frac{k_3\wedge k_4}{2}\right)+ \\ &(\eta_{\alpha\beta}\eta_{\gamma\delta}-\eta_{\alpha\gamma}\eta_{\beta\delta})\sin\left(\frac{k_1\wedge k_4}{2}\right)\sin\left(\frac{k_2\wedge k_3}{2}\right)+ \\ &(\eta_{\alpha\delta}\eta_{\beta\gamma}-\eta_{\alpha\beta}\eta_{\delta\gamma})\sin\left(\frac{k_3\wedge k_1}{2}\right)\sin\left(\frac{k_2\wedge k_4}{2}\right)\Big] \end{aligned} \tag{42}$$

where

$$p\wedge q = \theta^{\mu\nu}p_\mu q_\nu\ . \tag{43}$$

The vertices (41), (42) come from Fourier transforming A_μ, applying the plane wave product

$$e^{ipx}\star e^{iqx} = e^{\frac{i}{2}\theta^{\mu\nu}p_\mu q_\nu}e^{i(p+q)x} \tag{44}$$

and then using the standard QFT machinery of extracting vertices from Lagrangians. To fully quantize this theory, one needs to introduce the BRST terms to impose a gauge fixing condition. For the gauge

$$\partial_\mu A^\mu = \lambda\ , \qquad \lambda\in\mathbb{R} \tag{45}$$

appropriate gauge fixing action terms were found in [16].

The table is set and now one can quantize this action (e.g., in the path integral approach) and calculate correlation functions perturbatively in the coupling constant. What turns out is that gauge theories on the Moyal space (but also on other spaces reviewed in this seminar) exhibit interesting qualitetive properties. Due to the noncommutativity of the products, Feynman rules and diagrams naturally pick up phase factors depending on the noncommutativity θ and two momenta (analogous to the sines in cubic and quartic vertices). It turns out that for planar Feynman diagrams, these phases exactly cancel out and we are left with identical integrals as in commutative QFT, without any dependence on θ, and obtain UV divergences $|p|^n$ arising from the external momentum. In nonplanar diagrams the phase factor $\exp(ip\wedge k)$ remains and regularizes the UV behaviour with its oscillatory nature producing $\frac{1}{|p|^n}$ behaviour. This means that now the theory additionally misbehaves at the infrared sector, making it more difficult to renormalize such a theory, as new types of counterterms are needed. This phenomenon is called the UV/IR mixing in noncommutative geometry and it was heavily studied in various noncommutative spacetimes and various quantum theories, see e.g. [19–21].

On Moyal spacetime various ways at getting rid of UV/IR mixing were tested. In [22] a specific gauge fixing BF term was added to the action (37) which indeed does regularize the UV/IR behaviour, but, the formal commutative limit of this theory reduces to a scalar theory, not a Yang Mills gauge theory. Various other BF terms were tested in [23] which all require various constraints on the gauge field which survive the commutative limit and thus do not describe the photon. Another successful attempt [24] at curing UV/IR in Yang-Mills theories amounts to adding a harmonic oscillatory term of the form

$$S\supset\int(x^\rho\star_\theta x_\rho\star_\theta A_\mu)\star_\theta A^\mu \tag{46}$$

but such theory, although tame regarding UV/UR mixing, breaks gauge invariance.

The status so far is that there is just one type of "cure" for the UV/IR mixing on Moyal spacetime. It was recently achieved in [25–27] by employing the L_∞ algebra formalism in order to impose a braiding on the studied quantum theory. On the other hand, some authors [28, 29] argue that UV/IR mixing should be a fundamental property of the UV theory of quantum gravity and that noncommutative field theories provide a good arena to develop new renormalization tools for the full UV theory of quantum gravity.

Finally, (37) produces a finite tadpole diagram (1-point function), which is a good feature of this theory. Compared to many other models of gauge theories on noncommutative spacetimes, Moyal is one of the rare spaces where gauge theory has a stable vacuum.

IV. GAUGE THEORY ON $\mathbb{R}^3_\lambda$ SPACES

The space $\mathbb{R}^3_\lambda$ is defined as the deformed algebra of functions on $\mathbb{R}^3$ for which the coordinate functions close the $\mathfrak{su}(2)$ Lie algebra

$$\left[x^j,x^k\right]_\star = i\lambda\epsilon^{ijk}x^k\ . \tag{47}$$

The group $SU(2)$ satisfies all of the necessary assumptions from [1], so one can immediately perform the convolution algebra deformation from Section II.1. Unfortunately, the star product that is arrived at is extremely complicated and is expressed in terms of the BCH (Baker-Campbell-Hausdroff) function of the group $SU(2)$, making any field theoretic computations untractable. There exist other star products in literature [30–32] but they all share the technical problem that the star products are not cyclic in the integral

$$\int f \star_\lambda g \star_\lambda h \neq \int h \star_\lambda f \star_\lambda g \;, \tag{48}$$

which is essential for gauge invariance (for the same reasons already discussed in Moyal spacetime's Section III).

An alternative approach is to use a matrix basis of $\mathbb{R}^3_\lambda$ and define a type of tracial star product which will, by construction, obey tracial conditions. In [33, 34] it was shown that $\mathbb{R}^3_\lambda$ is realistically modeled by $L^2(SU(2))$, which, by Peter-Weyl theorem, is isomorphic to

$$\begin{aligned} L^2(SU(2)) &\cong \bigoplus_{j\in\mathbb{N}/2} V_{2j+1}\otimes V^*_{2j+1} \\ &\cong \bigoplus_{j\in\mathbb{N}/2} \mathrm{Mat}_{2j+1}(\mathbb{C}) \;, \end{aligned} \tag{49}$$

where V_{2j+1} are are the representation spaces of each half integer spin, labeled by eigenvalues of the Casimir operator C

$$C = (x^1)^2 + (x^2)^2 + (x^3)^2 \;. \tag{50}$$

In other words, when compared to the procedure from Section II.1, we do not lose information by not taking the inverse Fourier transform of the convolution product, it is enough to just consider functions $L^2(SU(2))$. The logic behind (49) is that we consider $\mathbb{R}^3_\lambda$ as the sum over all fuzzy spheres with radii $\lambda^2 j(j+1)$. With that said, we can now decompose, using (49), any $f \in L^2(SU(2)) \cong \mathbb{R}^3_\lambda$ using the basis functions/matrices of fuzzy spheres

$$\begin{aligned} f = \sum_{j=0}^{\infty}\sum_{n,m=-j}^{j} f^j_{n,m} v^j_{n,m} \;, \\ f^j_{n,m}\in\mathbb{C}\;,\; v^j_{n,m}\in \mathrm{Mat}_{2j+1}(\mathbb{C}) \;, \end{aligned} \tag{51}$$

with v^j-s being the basis matrices of $\mathrm{Mat}_{2j+1}(\mathbb{C})$ satisfying

$$\begin{aligned} v^j_{n,m} &\equiv |jn\rangle\langle jm| \;, \\ (v^j_{n,m})^\dagger = v^j_{n,m} \;, &\qquad v^{j_1}_{m,n} v^{j_2}_{q,p} = \delta^{j_1 j_2}\delta_{nq} v^{j_1}_{m,p} \;. \end{aligned} \tag{52}$$

Now we can define the star product simply as multiplication of functions (taking into account the matrix structure and relation (52)) and the integral of star products as the special trace

$$\int f_1 \star_\lambda ... \star_\lambda f_n d^3x := 8\pi\lambda^3 \sum_{j\in\mathbb{N}/2}(2j+1)\mathrm{Tr}_j(f^j_1 ... f^j_n) \;. \tag{53}$$

Finally, it is possible to express the three coordinate functions in the matrix basis as follows [33]

$$\begin{aligned} x^1 &= \frac{\lambda}{2}\sum_{j,m}\Big(\sqrt{(j+m)(j-m+1)}v^j_{m,m-1} + \\ &\qquad \sqrt{(j-m)(j+m+1)}v^j_{m,m+1}\Big) \;, \\ x^2 &= \frac{\lambda}{2i}\sum_{j,m}\Big(\sqrt{(j+m)(j-m+1)}v^j_{m,m-1} - \\ &\qquad \sqrt{(j-m)(j+m+1)}v^j_{m,m+1}\Big) \;, \\ x^3 &= \lambda\sum_{j,m} m\, v^j_{m,m} \end{aligned} \tag{54}$$

and they really do satisfy the $\mathfrak{su}(2)$ commutator rule (47).

IV.1. Gauge structure on $\mathbb{R}^3_\lambda$

Now let us define the gauge theoretic structure over $\mathbb{R}^3_\lambda$. Following [35], we consider the finitely generated (over $\mathcal{Z}(\mathbb{R}^3_\lambda)$) space of derivations

$$\mathcal{D} = \left\{ D_\alpha = \frac{i}{\lambda^2}\,[x^\alpha,\cdot]\,;\alpha = 1,2,3 \right\} \tag{55}$$

satisfying

$$[D_\alpha, D_\beta] = -\frac{1}{\lambda}\epsilon_{\alpha\beta\gamma} D_\gamma \;. \tag{56}$$

Now, just as before in the Moyal spacetime, any $U(1)$-like connection on $E = \mathcal{A}$

$$\begin{aligned} \nabla_\alpha(f) \equiv \nabla_{D_\alpha}(f) &= D_\alpha(f) + \nabla_\alpha(1)\star_\lambda f \\ &\equiv D_\alpha(f) + A_\mu \star_\lambda f \end{aligned} \tag{57}$$

is completely determined by its action on the identity element

$$1 = \sum_{j,m} v^j_{m,m} \;. \tag{58}$$

The expression for the curvature follows directly from (13)

$$\begin{aligned} F_{\alpha\beta} &\equiv F(D_\alpha, D_\beta) = [\nabla_\alpha, \nabla_\beta] - \nabla_{[D_\alpha, D_\beta]} \\ &= D_\alpha(A_\beta) - D_\beta(A_\alpha) + [A_\alpha, A_\beta]_\lambda + \frac{1}{\lambda}\epsilon_{\alpha\beta\gamma}A_\gamma \end{aligned} \tag{59}$$

and gauge transformations directly follow from (15), (16)

$$\begin{aligned} A^g_\alpha &= g^\dagger \star_\lambda A_\alpha \star_\lambda g + g^\dagger \star_\lambda D_\alpha(g) \;, \\ F^g_{\alpha\beta} &= g^\dagger \star_\lambda F_{\alpha\beta} \star_\lambda g \;. \end{aligned} \tag{60}$$

IV.2. Model of gauge theory on $\mathbb{R}^3_\lambda$

Gauge theories on $\mathbb{R}^3_\lambda$ exhibit unique properties even when compared to other noncommutative spacetimes.

Most notably, it was shown in [36] and [37] that a suitable quartic $\star_\lambda$-polynomial in A_α and $\partial_\beta A_\alpha$ which is gauge invariant, can be endowed with a harmonic term

$$S \supset \int (x_\mu \star_\lambda x^\mu) \star_\lambda \mathcal{P}(A) \; , \tag{61}$$

with $\mathcal{P}(A)$ another gauge invariant $\star_\lambda$-polynomial in A_α and $\partial_\beta A_\alpha$, such that the UV/IR mixing is under control. This idea was also proposed on Moyal spacetime [24] but it violated gauge symmetry, due to noncommutativity between coordinates and gauge transition functions. In $\mathbb{R}^3_\lambda$, on the other hand, $x_\mu \star_\lambda x^\mu$ is the Casimir operator and as such commutes with gauge transition functions. Even more, [36, 37] showed that this model's perturbative corrections to the 2-point and 4-point functions are perturbatively finite at all orders. Finally, this model [36, 37] has a finite 1-point function, meaning that its vacuum is stable in this sense. So all in all, $\mathbb{R}^3_\lambda$ admits a gauge theory with very good properties, crossing 3 of the major obstacles in constructing realistic gauge theories on noncommutative spaces. Unfortunately, generalizations of such models to 4 dimensions so far do not exist.

V. GAUGE THEORY ON κ–MINKOWSKI SPACETIME $\mathbb{R}^4_\kappa$

κ–Minkowski spacetime is defined by its commutation relation

$$[x^\mu, x^\nu]_\star = \frac{i}{\kappa}\left(a^\mu x^\nu - x^\mu a^\nu\right) \tag{62}$$

where a^μ is a constant 4–vector on Minkowski spacetime. Up to normalization, there are three classes of κ–Minkowski spacetimes

- timelike with a timelike a^μ, i.e., $a_\mu a^\mu = -1$,
- spacelike with a spacelike a^μ, i.e., $a_\mu a^\mu = 1$,
- lightlike with a lightlike a^μ satisfying $a_\mu a^\mu = 0$.

κ–Minkowski gained a lot of attention because it realizes doubly special relativity (DSR) [38, 39], in which the speed of light c as well as a minimal length scale l_p are invariant to a relativistic (quantum) group's action, as a concrete model [40]. Quantum field theories on timelike κ–Minkowski with star product obtained via the construction from Section II.1 were studied in [41–43].

Additionally, κ–Minkowski can be realized via Drinfeld twist deformation, using an Abelian twist from [44] or from a Jordanian twist [45], producing a $\star$ product more similar to the (33) than (31) due to its Hopf algebraic construction.

V.1. Gauge theory models on κ–Minkowski

For all three classes of κ-Minkowski, the Lie group $\mathcal{G}$ corresponding to coordinates Lie algebra can be written as

$$\mathcal{G} = \mathbb{R} \ltimes_J \mathbb{R}^3 \tag{63}$$

with J being a Jordan block defining the semidirect product. As such, all three Lie groups are non-unimodular. This implies [46], it turns out, that the $\star_\kappa$ product will always fail to satisfy the tracial property

$$\int f \star_\kappa g \star_\kappa h = \int h \star_\kappa f \star_\kappa g \tag{64}$$

when integrated with respect to the Lebesgue measure on $\mathbb{R}^4$. Some solutions to this problem were proposed:

1. Introducing a nontrivial integration measure function $\mu(x)$ for which

$$\int d^4x\, \mu(x)\, f \star_\kappa g \star_\kappa h = \int d^4x\, \mu(x)\, h \star_\kappa f \star_\kappa g \; . \tag{65}$$

   For this to happen, according to [47], $\mu(x)$ needs to satisfy

$$\partial_0 \mu(x) = 0 \; , \qquad x^i \partial_i \mu(x) = 3\mu(x) \; . \tag{66}$$

   The probles with this approach are that the $\mu(x)$ measure survives the commutative limit and that the measure function $\mu(x)$ destroys invariance under κ-Poincare quantum isometries.

2. By realizing the κ-Minkowski spacetime with the star product from [41, 42], the partial derivatives are actually no longer derivations of the deformed algebra of functions, but "twisted" derivations satisfying

$$\begin{aligned} \partial_0(f \star_\kappa g) &= \partial_0(f) \star_\kappa g + f \star_\kappa \partial_0(g) \\ \partial_i(f \star_\kappa g) &= \partial_i(f) \star_\kappa g + e^{\frac{i}{\kappa}\partial_0}(f) \star_\kappa \partial_i(g) \; . \end{aligned} \tag{67}$$

   It turns out that it is possible to twist the gauge structure from Section II.2 in a way to consistently define connections in the direction of twisted derivations. In this case, the gauge transformation rule for the curvature becomes

$$F^g_{\mu\nu} = g \star_\kappa \star F_{\mu\nu} \star_\kappa (e^{\frac{2i}{\kappa}\partial_0} g^\dagger) \tag{68}$$

   and for conjugated curvature

$$\left(F^\dagger_{\mu\nu}\right)^g = (e^{\frac{2i}{\kappa}\partial_0} g) \star_\kappa \star F^\dagger_{\mu\nu} \star_\kappa g^\dagger \; . \tag{69}$$

   Additionally, the failure of the tracial property is controlled by the same exponential twisting factor

$$\int d^n x f \star_\kappa g = \int d^n e^{-(n-1)\frac{i}{\kappa}\partial_0} g \star_\kappa f \tag{70}$$

   so for $n = 5$, i.e., for a 5-d κ–Minkowski spacetime, we get exact cancelation and the gauge invariance can be achieved only in $n = 5$. This restriction to $n = 5$ spacetime dimensions was discovered in [48].

Aside from this complication in achieving gauge invariance, UV/IR mixing is also present in quantum theories on κ–Minkowski [49] as well as a diverging tadpole diagram [50] in the 5-d κ–Minkowski $U(1)$ gauge theory. Additionally, in 4 dimensions, there is an additional problem in the 4–d κ–Minkowski spacetime which is that there does not exist a 4–dimensional differential calculus on this 4–dimensional spacetime [51, 52].

VI. CONCLUDING DISCUSSION

After multiple decades of development, gauge theory in noncommutative spacetimes is still riddled with the same complications, namely, UV/IR mixing, unstable vacuums or incorrect macroscopic spacetime dimensions. Even though the field is strongly physically motivated, it turns out that in practice, gauge theories face complicated problems. Recently, in [53], there was an interesting proposal to interpret the tadpole diagrams as contributions of a "thermal bath" arising from discreteness of time on (non-unimodular) noncommuative spaces, which might relax at least one of the major obstacles in noncommutative gauge theory.

Additionally, as a part of recent developments, the authors of [46] are currently working to utilize the general star product obtained in their paper to construct gauge theories in which noncommutativity of various Lie-algebraic types is codified in the structure constants. It was obtained (currently in the phase of writing a paper) that symmetry under NC gauge transformations and NC isometries can be expressed entirely in terms of the Lie-algebraic structure constants. The goal is to study Yang-Mills gauge theory on as many noncommutative spaces as possible and to see which spacetimes might overcome some of the hurdles discussed in this seminar paper. After some preliminary calculations, it seems that at least one of the candidate spaces will have a well behaving 1-point function for the photon field. Additionally, many candidate spaces are completely novel with coordinate Lie-algebras never explored before.

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