Introduction to Symplectic Geometry

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Symplectic geometry is the geometric arena in which calssical Hamiltonian mechanics takes place, i.e. **the geometry of phase space**. Let's therefore start by reviewing Hamiltonian mechanics.

1 A review of Hamiltonian mechanics (with an eye towards symplectic geometry).

Simple Hamiltonian systems are described by coordinates on a phase space:

$$\begin{aligned} &\text{Phase space: } M = U \times \mathbb{R}^n \qquad \text{with } U \subset \mathbb{R}^n \\ &\text{Coordinates: } (\underbrace{q^1, \dots, q^n}_{\substack{\text{generalized} \\ \text{coordinates} \\ (\text{on } U)}}, \underbrace{p_1, \dots, p_n}_{\substack{\text{momentum} \\ \text{coordinates}}}) \end{aligned}$$

Classical coordinates are smooth functions of $f : M \to \mathbb{R}$ or $f : M \to \mathbb{C}$. One such observable, namely the Hamlitonian $H \in C^\infty(M)$, plays a particularly important role since it determines the dynamics of the system via Hamilton's equations of motion:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

E.g.: For

$$H(q,p) = \frac{n}{2} \sum_{\substack{i=1 \\ j=1}}^{n} \left(p_i p_j \delta^{ij} + q^i q^j \delta_i j \right),$$

we set the harmonic oscillator equations of motion:

$$\dot{q}^i = p_i, \quad \dot{p}_i = -\dot{q}^i \implies \ddot{q}^i + q^i = 0.$$

Let's rewrite Hamiltonian's equations of motion in a more geometric form, namely as flow equations determined by a vector field on the phase space M. To do this, we collect q^i and p_i in a vector:

$$z = \begin{pmatrix} q^i \\ p_i \end{pmatrix}$$

and introduce the vector field $X_H(z)$ on M as follows:

$$X_H(z) = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q^i} \end{pmatrix}$$

Hamilton's equations of motion then take the form of the flow equations

$$\dot{z} = X_H(z) \qquad \left(\Longleftrightarrow \begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial q^i} \end{pmatrix} \right)$$

The vector field $X_H(z)$ looks a bit like the gradient of H, but is not exactly equal to the gradient. Clearly however, there is a relation between X_H and the gradient of H. To see what this relation is, we introduce the following 2-form on M:

$\omega = dq^i \wedge p_i$ (the Einstein summation convention is used, i.e. the sum over i is understood)

Let's consider then the vector field X_H, written out like a pro-differential geometer would do it (namely like a differential operator on M):

$$X_H = \frac{\partial H}{\partial q^i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

and an arbitrary vector field

$$V = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}.$$

We then have that

$$\begin{aligned} \omega(X_H, V) &= (dq^i \wedge dp_i)\left(\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}, a^k \frac{\partial}{\partial q^k} + b_k \frac{\partial}{\partial p_k}\right) \\ &= \frac{\partial H}{\partial p_i} b_i - \left(-\frac{\partial H}{\partial q^i}\right) a^i = \frac{\partial H}{\partial p_i} b_i + \frac{\partial H}{\partial q^i} a^i \end{aligned}$$

Compare that with the action of the gradient 1-form dH on V

$$\begin{aligned} dH(V) &= \left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i\right)\left(a^j \frac{\partial}{\partial q^j} + b_j \frac{\partial}{\partial p_j}\right) \\ &= \frac{\partial H}{\partial q^i} a^i + \frac{\partial H}{\partial p_i} b_i \end{aligned}$$

We thus see that

$$\boxed{\omega(X_H, V) = dH(V) \qquad \text{for all vector fields } V}$$

The relation between X_H and the gradient one-form dH is thus determined by the 2-form ω. In fact, given ω and H, we can construct X_H and thus write down Hamilton's equations of motion. Tho see why this is the case, let us write down $\omega(X_H, V) = dH(V)$ in matrix form. We introduce the matrix Ω associated to ω as

$$\Omega = \begin{pmatrix} \omega\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right) & \omega\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_j}\right) \\ \omega\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^j}\right) & \omega\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) \end{pmatrix} = \begin{pmatrix} 0 & \delta_i^j \\ -\delta_j^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

Calling $X = x^i \frac{\partial}{\partial q^i} + \varphi_i \frac{\partial}{\partial p_i}$ and $V = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$, the equation $\omega(X, V) = dH(V)$ can then be written in matrix form as

$$(x^i, \varphi_i) \begin{pmatrix} 0 & \delta_i^j \\ -\delta_j^i & 0 \end{pmatrix} \begin{pmatrix} a_j \\ b^j \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial q^j} & \frac{\partial H}{\partial p_j} \end{pmatrix} \begin{pmatrix} a^j & b_J \end{pmatrix}$$

If a^i, b_i are arbitrary, we can write

$$\begin{pmatrix} x^i & \varphi_i \end{pmatrix} \begin{pmatrix} 0 & \delta_i^j \\ -\delta_j^i & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial q^j} & \frac{\partial H}{\partial p_j} \end{pmatrix}$$

Since the matrix $\begin{pmatrix} 0 & \delta_i^j \\ -\delta_j^i & 0 \end{pmatrix}$ is invertible, this equation can be solved for x^i, φ_i. The invertibility of the matrix Ω thus implies that the equation

$$\omega(X_H, V) = dH(V) \qquad \text{for all vector field } V$$

determines the vector field X_H uniquely (in terms of the components of the gradient dH and the matrix elements of Ω).

Clearly, the 2-form ω plays an important role in writing down Hamilton's equations of motion in a geometricc way. And it is important that it is non-degenerate in the sense that its matrix is invertible! Note also that its matrix is constant. This doesn't look important now, but it is a useful property to have if we want to relate ω to another useful ingredient of Hamiltionan mechanics, namely the Poisson bracket.

The Poisson bracket $\{F, G\}$ of 2 functions $F, G \in C^\infty(M)$ on the phase space is introduced as

$$\{F, G\} = \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i}$$

Hamilton's equations of motion thate the following alternative form

$$\dot{q}^i = \{q^i, H\}, \quad \dot{p}_i = \{p_i, H\}$$

and the time evolution of a function $F(q^i(t), p_i(t))$ (with $q^i(t)$, $p_i(t)$ solutions of Hamilton's equations of motion) is given by

$$\begin{aligned} \dot{F} &= \frac{d}{dt} F(q^i(t), p_i(t)) = \frac{\partial F}{\partial q^i}(t) \dot{q}^i(t) + \frac{\partial F}{\partial p_i} \dot{p}_i(t) \\ &\qquad = \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} = \{F, H\} \\ \Longrightarrow \; \dot{F} &= \{F, H\} \end{aligned}$$

Note in particular that an observable $F \in C^\infty(M)$ is a constant of motion (a so-called "first integral") if and only if it Poisson-commutes with the hamiltonian H: $\{F, H\} = 0$.

The Poisson bracket can also be written in termso of $\omega = dq^i \wedge dp_i$ as

$$\{F, G\} = \omega(X_F, X_G)$$

with $\omega(X_F, V) = dF(V)$, $\omega(X_G, V) = dG(V)$ for every field V. The Poisson bracket is thus not a "new" ingredient in Hamiltonian mechanics but one that can be defined in termso of ω!

Indeed:

$$\omega(X_F, X_G) = dF(X_G) = \left(\frac{\partial F}{\partial q^i}dq^i + \frac{\partial F}{\partial p_i}dp_i\right)\left(\frac{\partial G}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial G}{\partial q^j}\frac{\partial}{\partial p_j}\right)$$
$$= \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} = \{F, G\}$$

Although it is usually not stressed, the fact that ω is constant is important for some properties of the Poisson bracket. In particular, it ensures that the Poisson bracket obeys the Jacobi identity:

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0 \qquad \forall F, G, H \in C^\infty(M)$$

So far, we assumed that phase space is of the form $U \times \mathbb{R}^n$, with $U \subset \mathbb{R}^n$. This is often too restrictive. Take for instance a particle that is constrained to move on a circle or a sphere, in which case it is not true that $U \subset \mathbb{R}^n$. Moreover, often when solving a mechanical problem, one uses symmetries and constants of motion to reduce the number of degrees of freedom. In that case, the phase space of the system with a reduced number of degrees of freedom is often no longer of the form $U \times \mathbb{R}^n$.

Example. For a Hamiltonian dynamical system, the Hamiltonian H is conserved, so the motion takes place in a hypersurface of phase space, described by the equation $H(q^i, p_i) = E = \text{constant}$. If $i = 1, \ldots, n$, this hypersurface is $(2n-1)$-dimensional and corresponds to the subspace $\Sigma_i = H^{-1}(E)$ of phase space. For fixed E, we can thus restrict ourselves to Σ_i. Since Σ_i is $(2n-1)$-dimensional, it is not a phase space (which should be even-dimensional) itself. To turn Σ_i into a phase space, we note that a phase space can be viewed as a set of solutions of the equations of motion (since usually the solutions are in 1-to-1 correspondence with the possible initial conditions). Since, for fixed E, all solutions lie within Σ_i, we should really identify phase space as Σ_i modded out by the equivalence relation $\sim$, which considers 2 points of Σ_i equivalent if they lie on the same solution of the equations of motion. The quotient space $\Sigma_i/\sim$ is then $(2n-2)$-dimensional, as it should be for a phase space. It is non-trivial, but can be shown that $\Sigma_i/\sim$ is indeed a phase space of dimension $2(n-1)$ (so with $n-1$ degrees of freedom instead of the n one started from). The non-trivial aspect of the preceding sentence consists in showing that $\Sigma_i/\sim$ has a natural symplectic form and this is non-trivial because $\Sigma_i/\sim$ is no longer of the simple form $U \times \mathbb{R}^n$.

We should thus broaden our notion of phase space to more general manifolds. In order to define Hamiltonian dynamics on it and in particular to define an analogue of the vector field X_H associated to a Hamiltonian H, we equip our manifolds with a suitable generalization of the 2-form ω. It turns out that the following definition captures the correct generalization.

Definition A symplectic form on a manifold M is a 2-form $\omega \in \Lambda^2(M)$, whicch is:
1) non -degenerate, i.e. if $\omega(x, \varphi) = 0$ $\forall$ vector fields φ, then $x = 0$
2) closed, i.e. $d\omega = 0$.

The non-degeneracy of ω ensures that we can define a flow vector field X_H associated to a Hamiltonian H, so that we can discuss Hamiltonian dynamics on a manifold equipped with a symplectic form. The closedness of ω is a suitable generalization of the constants of $dq^i \wedge dp_i$ on $U \times \mathbb{R}^n$ and will help in defining a Poisson bracket that has nice properties (like the Jacobi identity).

Introducing a tangent space basis $\left\{\frac{\partial}{\partial x^\mu}\right\}$ (with respect to local coordinates (x^μ) on M), the non-degeneracy of the ω means that the matrix

$$\omega_{\mu\nu} \equiv \omega\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$$

is non-degenerate. Note that then (with $d = \dim(M)$)

$$\det(\omega_{\mu\nu}) = \det((\omega_{\mu\nu})^{\mathsf{T}}) = \det(-\omega_{\mu\nu}) = (-1)^d \det(\omega_{\mu\nu})$$

So one has that $(-1)^d = 1$ and d, the dimension of the manifold M that ω lives on, is necessarily even!

We can then define the notion of a symplectic manifold:

Definition A symplectic manifold (M, ω) is a manifold M, together with a symplectic form ω.

Above, we say that symplectic manifolds are necessarily even-dimensional.

In mathematics, once one introduces a new notion of structure on spaces, one should also consider maps that preserve this structure. These maps are called symplectomorphisms.

Definition Given 2 symplectic manifolds (M, ω) and (M', ω'), a symplectomorphism is a diffeomorphism $\Phi : M \to M'$ that satisfies

$$\Phi^* \omega' = \omega$$

where Φ^* denotes the pullback of Φ.

In local coordinates:

$$\Phi : M \to M' \quad : \quad x^\mu \to y^\mu(x^\mu)$$
$$(\Phi^* \omega')_{\mu\nu} = \omega'_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\mu}$$

so the condition $\Phi^* \omega' = \omega$ means that

$$\omega'_{\alpha\beta} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}$$

(Note that $\sim$ symplectomorphism from (M, ω) to itself is a diffeomorphism $\Phi : M \to M$, $x^\mu \to \tilde{x}^\mu(x^\nu)$ such that $\omega_{\rho\sigma} \frac{\partial x^\rho}{\partial x^\mu} \frac{\partial x^\sigma}{\partial x^\nu} = \omega_{\mu\nu}$. Compare this to the definition of an isometry in Riemannian geometry!)

Remark No sensibe physicist utters the word "symplectomorphism". Instead, physicists use the term "canonical transformation".

As an aside on the comparison between symplectic geometry and Riemannian geometry, we note that the 2 are quite different. With the 2-form ω one cannot compute lengths (those can be computed with the metric in Riemannian geometry). One can however integrate ω over 2-surfaces, i.e. one can compute a notion of area. For a symplectic manifold of dimension $d = 2n$, we also have that $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$ is (proportional to) a volume form, so we can also compute volumes in symplectic geometry. Below, we will see another striking difference between symplectic and Riemannian geometry (namely Darboux' theorem).

Let us give an example of a symplectic manifold, namely the example in which the phase space $U \times \mathbb{R}^n$ (with $U \in \mathbb{R}^n$) is generalized in the sense that the configuration space U is replaced by an arbitrary manifold Q. This is typical for the kind of phase spaces that one encounters in classical mechanics.

Example 1: The contangent bundle. Let Q be a manifold of dimension n. The cotangent bundle T^*Q of Q is (as a set) given by

$$T^*Q \equiv \bigcup_{m\in Q} T^*_m Q$$

where $T^*_mQ = \{\text{space of 1-forms at } m \in Q\} = \{\text{space of linear maps } T_mQ \to R\}$. This bundle comes with a natural projection map

$$\tau^* : T^*Q \to Q : \ \tau^*(T^*_mQ) = \{m\}.$$

(τ^* maps are an element (m, λ) of T^*Q (with $m \in Q$ and $\lambda \in T^*_mQ$) to m.)

To introduce local coordinates on T^*Q, we first equip Q with local coordinated $\{q^i\}$, $i = 1, \ldots, n$. A 1-form in T^*_mQ is then of the form $p_i dq^i$, so we use the $\{p_i\}$ to parametrize the space of all 1-form T^*_mQ. We can thus use the following local coordinates on T^*Q

$$(q^1, \ldots, q^n, p_1, \ldots, p_n)$$

Note that this choice of coordinates is rather special. The p_i are induced by the choice of coordinates $\{q^i\}$ on Q, i.e. the p_i refer to the 1-form $p_i dq^i \in T^*_mQ$, written in the coordinates $\{q^i\}$ on Q.

Note that if one has 2 regions $U \subset T^*Q$ and $V \subset T^*Q$ with $U \cap V \neq \emptyset$ and with coordinates on U given by (q^i, p_i) and on V given by $(\tilde{q}^i, \tilde{p}_i)$, these coordinates should on the overlap $U \cap V$ be related with

$$(1)\quad \begin{cases} \tilde{q}^i = \tilde{q}^i \quad \text{(a smooth map)} \\ \tilde{p}_i = p_j \frac{\partial q^j}{\partial \tilde{q}^i} \quad \text{(so that } \tilde{p}_i d\tilde{q}^i = p_i dq^i\text{, or } p_i \text{ transforms as a co-vector)} \end{cases}$$

Such a coordinate transformation in T^*Q is in particular smooth, indicating that T^*Q is a smooth manifold (if Q is).

T^*Q is also a symplectic manifold. To show this, we first introduce the so-called "Lioville 1-form" or "tautological 1-form" or "symplectic potential". Thi is a '-form on T^*Q that is in local coordinates given by

$$\theta = p_i dq^i$$

Note A generic 1-form on T^*Q is an object of the form $\alpha_i(q,p)dq^i + \beta^i(q,p)dp_i$. For θ, one has $\alpha_i = p_i$ and $\beta^i = q^i$. This should be considered with a 1-form of Q, which is of the form $\alpha_i(q)dq^i$
$\downarrow$
only dependence on Q!

Even though ew wrote down θ in local coordinates (q^i, p_i), the precise choice of coordinates does not matter that much. E.g. taking 2 sets of coordinates (q^i, p_i), $(\tilde{q}^i, \tilde{p}_i)$ related via (1), one has that

$$\tilde{p}_i d\tilde{q}^i = \left(p_j \frac{\partial q^j}{\partial \tilde{q}^i}\right)\left(\frac{\partial \tilde{q}^i}{\partial q^k} dq^k\right) = p_j \left(\frac{\partial q^j}{\partial \tilde{q}^i}\frac{\partial \tilde{q}^i}{\partial q^k}\right) dq^k = p_i dq^i$$

The one-form θ thus assumes some form in each set of coordinates (q^i, p_i), $(\tilde{q}^i, \tilde{p}_i)$ related via (1). This indicates that θ is globally well-defined (in a coordinate independent manner).

Using θ, we can define a symplectic form ω as follows

$$\omega = -d\theta = dq^i \wedge dp_i$$

.

With respect to the local coordinate basis $\left\{\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i}\right\}$, the matrix of ω is $\begin{pmatrix} 0 & \chi \\ -\chi & 0 \end{pmatrix}$, so ω is non-degenerate. Moreover, ω is also closed, since

$$d\omega = -d^2\theta = 0.$$

So, (T^*Q, ω) is a symplectic manifold (often called “momentum phase space”).

Note that $d\omega = 0$ implies that ω is locally exact (via the $\boxdot$ lemma), so that locally one can always find a one-forem θ with $\omega = -d\theta$. The above example is special, because θ is not only locally defined, but also globally. So for the co-tangent bundle T^*Q, ω is not only locally exact, but it is actually (globally) exact. This is however special and not a generic feature of symplectic manifolds. As a counterexample consider:

Example 2 Any compact 2-dimensional surface without boundary (e.g. a sphere, torus, genus-g surface,...) is symplectic, but ω is not (globally) exact.

Indeed, we can just take the volume-form for ω, which is non-degenerate and closed (since $d\omega = 0$, as $d\omega$ is a 3-form on a 2-dimensional manifold). ω is not exact, i.e. there is no globally well-defined one-form θ, such that $\omega = -d\theta$, however. If ω were exact, we would have

$$\text{Volume } = \int_M \omega = -\int_M d\theta \underset{\substack{\downarrow \\ \text{Stokes}}}{=} -\int_{\partial M} \theta = 0 \qquad (\partial M \neq \emptyset)$$

i.e. the volume of the 2-surface would be zero, a contradiction!

We see that for the cotangent bundle, the symplectic form locally takes the same form as what we had for $U \times \mathbb{R}^n$.

$$\omega = dq^i \wedge dp_i$$

This is a general result in symplectic geometry, and is the content of the following theorem (without proof).

Theorem (Darboux) Every point $m \in M$ of a symplectic manifold (M, ω) has an open neighborhood $U \subset M$ and a chart with local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ such that in these coordinates

$$\omega|_U = dq^i \wedge dp_i$$

The (q^i, p_i) are called (local) canonical coordinates.

Darboux' theorem provides one of the most striking differences between symplectic and Riemannian geometry. In Riemannian geometry, one can find coordinates such that the metric $g_{\mu\nu}$ is the flat one $\eta_{\mu\nu}$ in a point. If the manifold has curvature, this can however not be done in an entire neighborhood of that point. Darboux' theorem states that in symplectic geometry one can choose the equivalent of "flat" (canonical) coordinates in a neighborhood of a point, i.e. coordinates in which $\omega_{\mu\nu} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$.

In this sense, Darboux' theorem can be viewed as saying that there is no notion of curvature in symplectic geometry.

On a symplectic manifold (M, ω) we can discuss Hamiltonian dynamics in pretty much the same way as we did it before. We use a function $H \in C^\infty(M)$ and the symplectic form ω to define a Hamiltonian vector field X_H via

$$\omega(X_M, \varphi) = dH(\varphi) \quad \forall \text{ vector fields}$$

In local canonical coordinates (q^i, p_i), we have as before that

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

Using X_H, one can then consider Hamilton's equations of motion in local coordinates $z = (q^i, p_i)$ as

$$\dot{z} = X_H \quad \Longleftrightarrow \quad \dot{q}^i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

The Poisson bracket is also defined in terms of ω as before.

Definition The Poisson bracket $\{F, G\}$ of 2 functions $F, G \in C^\infty(M)$ is defined by

$$\{F, G\} = \omega(X_F, X_G)$$

$$\begin{aligned} \text{with } \omega(X_F, \varphi) &= dF(\varphi) \\ \omega(X_F, \varphi) &= dF(\varphi) \qquad \forall \text{ vector fields } \varphi \end{aligned}$$

It satisfies the following properties:

1) $\{\cdot, \cdot\}$ is bilinear over $\mathbb{R}$.

2) anti-symmetr: $\{F, G\} = -\{G, F\}$

3) Jacobi identity: $\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0$

4) $\{F, GH\} = G\{F, H\} + H\{F, G\}$ (one says that $\{F, \cdot\}$ is a derivation on $C^\infty(M)$)

5) $\{F, G\} = 0$ for all $F \in C^\infty(M) \iff dG = 0$

Let us by way of example prove the Jacobi identity, using the formalism of symplectic geometry. We start from $d\omega = 0$:

$$\begin{aligned} 0 =& d\omega(X_F, X_G, X_H) \\ =& X_F(\omega(X_G, X_H)) - X_G(\omega(X_F, X_H)) + X_X(\omega(X_F, X_G)) \\ & - \omega([X_F, X_G], X_H) + \omega([X_F, X_H], X_G) - \omega([X_G, X_H], X_F) \\ & (\text{with } [X_F, X_G]f = X_F(X_G(f)) - X_G(X_F(f))) \end{aligned}$$

This is just the definition of $d\omega(X_F, X_G, X_H)$ written out in a coordinate-independent way.

Note that for a generic function $G \in C^\infty(M)$, we have

$$X_F G = dG(X_F) = \omega(X_G, X_F) = \{G, F\}$$

so $X_F(\omega(X_G, X_H)) = \{\omega(X_G, X_H), F\} = \{\{G, H\}, F\} \dots \{F, \{G, H\}\}$.

Likewise we have

$$\begin{aligned}
-\omega([X_F, X_G], X_H) = +\omega(X_H, [X_F, X_G]) = [X_F, X_G]H &= X_F(X_G(H)) - X_G(X_H(H)) \\
&= X_F\{H, G\} - X_G\{H, F\} \\
&= \{\{H, G\}, F\} - \{\{H, F\}, G\} \\
&= \{F, \{G, H\}\} + \{G, \{H, F\}\}
\end{aligned}$$

We then get

$$\begin{aligned}
0 =& X_F(\omega(X_G, X_H)) - \omega([X_F, X_G], X_H) + X_G(\omega(X_H, X_F)) - \omega([X_G, X_H], X_F) \\
&+ X_H(\omega(X_F, X_G)) - \omega([X_H, X_F], X_G) \\
=& -\{F, \{G, H\}\} + \{F, \{G, H\}\} + \{G, \{H, F\}\} + \text{ cyclic permutations } (F \to G \to H) \\
=& \{G, \{H, F\}\} + \{F, \{G, H\}\} + \{H, \{G, F\}\}
\end{aligned}$$

This is just the Jacobi identity.

Let us finish with an example of Hamiltonian dynamics, that is most easily formulated using a symplectic form that is **not** the standard canonical one. This is the example of a particle in a magnetic field. Since a magnetic field does no work, we expect that the Hamiltonian of a particle in a magnetic field is the usual free particle one

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2).$$

How can we then incorporate the effect of the magnetic field? The answer is by not working with the standard canonical symplectic form. Instead of taking

$$\omega_0 = dq^i \wedge dp_i \qquad (i = 1, 2, 3)$$

we take instead

$$\omega = \omega_0 + \frac{n}{2} F_{ij}(q) dq^i \wedge dq^j = dq^i \wedge dp_i + \frac{n}{2} F_{ij}(q) dq^i \wedge dq^j$$

where $F_{ij} = \partial_i A_j - \partial_j A_i = -\varepsilon_{ijk} B_k$ (with A_i the vector potential and B_i the magnetic field).

Then

$$\begin{aligned}
X_H &= \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_j} F_{ji} \frac{\partial}{\partial p_i} \\
&= \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \left(-\frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j} F_{ji}\right)\frac{\partial}{\partial p_i}
\end{aligned}$$

Let us chech this. Pick $\varphi = a^i\frac{\partial}{\partial q^i} + b_i\frac{\partial}{\partial p_i}$

$$
\begin{aligned}
\omega(X_H,\varphi) =& dq^i\wedge dp_i\left[\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^j} + \left(-\frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial p_l}F_{lj}\right)\frac{\partial}{\partial p_j}, a^k\frac{\partial}{\partial q^k} + b_k\frac{\partial}{\partial p_k}\right] \\
&+\frac{1}{2}F_{ij}dq^i\wedge dq^j\left[\frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} + \left(-\frac{\partial H}{\partial q^k} + \frac{\partial H}{\partial p_l}F_{lk}\right)\frac{\partial}{\partial p_l}, a^m\frac{\partial}{\partial q^m} + b_m\frac{\partial}{\partial p_m}\right] \\
=&\frac{\partial H}{\partial p_i}b_i - \left(-\frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_l}F_{li}\right)a^i + \frac{1}{2}F_{ij}\left(\frac{\partial H}{\partial p_i}a^j - \frac{\partial H}{\partial p_j}a^i\right) \\
=&\frac{\partial H}{\partial p_i}b_i + \frac{\partial H}{\partial q^i}a^i - \cancel{\frac{\partial H}{\partial p_l}F_{li}a^i} + \cancel{\frac{\partial H}{\partial p_l}F_{li}a^i} \\
=&\frac{\partial H}{\partial p_i}b_i + \frac{\partial H}{\partial q^i}a^i \\
=&dH(\varphi) \quad \checkmark
\end{aligned}
$$

Note that ω is a symplectic form. In matrix for it is given by $\begin{pmatrix} F_{ij} & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$ and this is invertible, so ω is non-degenerate. Moreover:

$$
\begin{aligned}
d\omega &= \underbrace{d\omega_0}_{=0} + \frac{1}{2}\partial_k F_{ij}(q)dq^k\wedge dq^i\wedge dq^j \\
&= \frac{1}{2}\partial_k(\partial_i A_j - \partial_j A_i)dq^k\wedge dq^i\wedge dq^j = \partial_k\partial_i A_j dq^k\wedge dq^i\wedge dq^j \\
&= 0
\end{aligned}
$$

(since $\partial_k\partial_i$ is symmetric in (i,j), while $dq^k\wedge dq^i$ is anti-symmetric in $[i,j]$)

The equation $\dot{z} = X_H$ in local coordinates (q^i, p_i) reads

$$
\begin{pmatrix} \dot{q}^i \\ \dot{p}_j \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j}F_{ji} \end{pmatrix}.
$$

With $H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2)$, this reduces to

$$
\begin{pmatrix} \dot{q}^i \\ \dot{p}_j \end{pmatrix} = \begin{pmatrix} \frac{p_i}{m} \\ \frac{p_j}{m}F_{ji} \end{pmatrix} = \begin{pmatrix} \frac{p_i}{m} \\ -\varepsilon_{ijk}\frac{p_j}{m}B_k \end{pmatrix}
$$

$$
\Longrightarrow \begin{cases} p_i = m\dot{q}_i \\ \dot{p}_i = \varepsilon_{ijk}\frac{p_j}{m}B_k = \varepsilon_{ijk}v_j B_k \end{cases}
$$

These indeed give rise to the correct equations of motion of a particle in a magnetic field (i.e. a particle subject to the Lorentz force).

Finally, let us note that even though

$$
\omega = dq^i\wedge dp_i = \frac{1}{2}F_{ij}(q)dq^i\wedge dq^j
$$

is not in the standard canonical form, Darboux' theorem states that locally we should be able to bring it in canonical form. Let us show that this is indeed the case, by using that locally we have

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

So:

$$\begin{aligned}
\omega &= dq^i \wedge dp_i + \partial_i A_j dq^i \wedge dq^j \\
&= dq^i \wedge dp_i + d(A_i dq^i) \\
&= dq^i \wedge dp_i + dA_i \wedge dq^i \\
&= dq^i \wedge dp_i - dq^i \wedge dA_i \\
&= dq^i \wedge d(p_i - A_i)
\end{aligned}$$

So, we see that

$$\begin{aligned}
\tilde{q}^i &= q^i \\
\tilde{p}_i &= p_i - A_i
\end{aligned}$$

are local Darboux canonical coordinates for which $\omega = d\tilde{q}^i \wedge d\tilde{p}_i$. Note however that usually A_i can not be found globally, so the Darboux coordinates $(\tilde{q}^i, \tilde{p}_i)$ can typically only be defined in a local patch.